

# Pseudo-Finsler spaces modeled on a pseudo-Minkowski space

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## Abstract

We adopt a vierbein formalism to study pseudo-Finsler spaces modeled on a pseudo-Minkowski space. We show that it is possible to obtain closed expressions for most tensors in the theory, including Berwald, Landsberg, Douglas, non-linear connection, Ricci scalar. These expressions are particularly convenient in computations since they factor the dependence on the base and the fiber. Finally, the paper contains some comments on the geometry of Berwald spaces.

*Keywords:*

Berwald space, Minkowski space, Ricci scalar.

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## 1. Introduction

In this work by pseudo-Minkowski space we mean a pair  $(V, L)$  where  $V$  is a  $m$ -dimensional vector space and  $L: \bar{\omega} \rightarrow \mathbb{R}$  is a function defined on the closure of an open cone<sup>1</sup>  $\omega \subset V$  with vertex the origin, such that

- (a)  $L \in C^4(\omega) \cap C^0(\bar{\omega})$ ,
- (b)  $\forall s > 0$ , and  $y \in V$  we have  $L(sy) = s^2 L(y)$ ,

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<sup>1</sup>We do not assume it to be convex.

(c) the Hessian  $g := (\partial_\mu \partial_\nu L) dy^\mu \otimes dy^\nu$  is non-degenerate on  $\omega$ .

It is understood that the Hessian is calculated using any set of linear coordinates on  $V$  (namely a dual basis on  $V^*$ ). More specific theories are possible, for instance if  $g$  is positive definite and  $\omega = V \setminus 0$  one speaks of Minkowski space, while if  $g$  has Lorentzian signature and  $\omega$  is a sharp convex cone one speaks of Lorentz-Minkowski space. We warn the reader that already in the positive definite case the concept of Minkowski space is not homogeneously defined in the literature (compare [1, 2]).

If we were concerned with just Minkowski and Lorentz-Minkowski spaces we would probably add the condition

(d)  $L|_\omega \neq 0$  and  $L|_{\partial\omega} = 0$ ,

in the definition. However, we shall not use it in our derivations (so vector spaces endowed with a non-degenerate bilinear form are included in the definition).

A pseudo-Finsler space is instead a pair  $(M, \mathcal{L})$  where  $M$  is a  $m$ -dimensional manifold and  $\mathcal{L} : \bar{\Omega} \rightarrow \mathbb{R}$ ,  $\Omega \subset TM$ ,  $\pi(\Omega) = M$ , with the property that if we define  $\Omega_x := \pi^{-1}(x) \cap \Omega$ ,  $\mathcal{L}_x := \mathcal{L}|_{\Omega_x}$ ,  $\Omega_x \subset T_x M$ , then the pair  $(T_x M, \mathcal{L}_x)$  is a pseudo-Minkowski space with a signature independent of  $x$ . One can assume various differentiability conditions on the dependence of  $\mathcal{L}$  on  $x$ , for our calculations  $C^2$  will suffice. Of course, one could define Finsler and Lorentz-Finsler spaces adding the conditions introduced above for pseudo-Minkowski spaces.

This paper is concerned with a special type of pseudo-Finsler space, namely those for which all pseudo-Minkowskian tangent spaces  $(T_x M, \mathcal{L}_x)$  are modeled on the same pseudo-Minkowski space  $(V, L)$ . This means that locally we can find a linear isomorphisms (sufficiently differentiable in  $x \in U \subset M$ )

$$\varphi_x : T_x M \rightarrow V, \tag{1}$$

such that  $\varphi_x(\Omega_x) = \omega$ , and  $L(\varphi_x(y)) = \mathcal{L}_x(y)$  for every  $y \in \bar{\Omega}_x$ . Parallelizable manifolds admit structures of this type and parallelizability can be relaxed provided  $(V, L)$  admits a Lie group  $G$  of linear isomorphisms (they preserve  $\mathcal{L}$  and its domain). In that case it is sufficient that  $M$  admits a  $G$ -structure [3]. Our considerations will be of local character so we shall not enter into a more detailed discussion [3]. The idea of Finsler space modeled on the same Minkowski space has been first introduced and investigated by Ichijyo [3] and

further progress has been obtained by Aikou in [4]. Matsumoto [5], Izumi [6], Sakaguchi [7], and Asanov and Kirnasov [8] had also considered similar spaces referring to them as *1-form Finsler spaces*. Szilasi and Tamásy call them *affine deformations of Minkowski spaces* [9], while Libing Huang, in a recent work with a goal similar to our own, calls them *single colored Finsler spaces* [10].

Roughly speaking, while in pseudo-Finsler spaces the anisotropic features change from point to point and can even be absent in some open subset of  $M$ , in pseudo-Finsler spaces modeled on a pseudo-Minkowski space the anisotropic features do not depend on the point. Of course, pseudo-Riemannian manifolds provide an example of pseudo-Finsler space modeled on a pseudo-Minkowski space. Berwald spaces are also modeled on the same Minkowski space. The linear isometry between different tangent spaces is provided by the non-linear (actually linear) parallel transport connecting the two points.

However, Berwald spaces are quite rigid and so perhaps not that interesting. In fact, let  $G_x$  be the group of linear isomorphisms of  $(T_x M, \mathcal{L}_x)$  and let  $\mathfrak{g}_x$  be its Lie algebra. Let  $R_x$  be the curvature of the Berwald connection, which is a two form with values in the endomorphisms of  $T_x M$ . We have (for related ideas cf. [11])

**Theorem 1.1.** *Let  $(M, \mathcal{L})$  be a Berwald pseudo-Finsler space. The Lie algebra generated by  $R_x$  is contained in  $\mathfrak{g}_x$ . If for every  $x \in M$ ,  $G_x$  is just the identity (e.g. because it is modeled on the same pseudo-Minkowski space with no symmetries) then the Berwald pseudo-Finsler space is really a pseudo-Minkowski space.*

*Proof.* Since the space is Berwald the transport with the Berwald connection is a linear isomorphism of tangent pseudo-Minkowski spaces (it preserves also the Finsler Lagrangian [12, Prop. 10.1.1][13, Prop. 2.1]). Let  $X, Y \in T_x M$ . By the Ambrose-Singer theorem  $R_x(X, Y)$  is an element of the Lie Algebra of  $\text{Hol}_x$ , the holonomy Lie group of the connection at  $x$ . Thus  $t \mapsto \exp[tR_x(X, Y)]$  is a one parameter group of linear isomorphisms of  $(T_x M, \mathcal{L}_x)$ , hence  $R_x(X, Y) \in \mathfrak{g}_x$ .

If  $G_x$  is trivial then  $R = 0$  and since the Berwald connection is torsionless it follows that the space is locally pseudo-Minkowski (i.e.  $M$  is covered by patches over which  $\mathcal{L}$  has no dependence on  $x$ ), see e.g. [14], hence modeled on the same pseudo-Minkowski space. Actually, it is pseudo-Minkowski since the transition map in the intersection of two patches must belong to  $G$ , the

linear group of isomorphisms of the model pseudo-Minkowski space, which is trivial.  $\square$

This result shows that Berwald spaces modeled on a pseudo-Minkowski space can be non-trivial only if the model pseudo-Minkowski space has symmetries, a fact which limits considerably the family of these spaces.

In recent years there has been a growing interest in Finsler geometry. Unfortunately, in applications the calculations are often prohibitive so it is natural to look for a smaller family of spaces for which they become amenable.

In this work we shall be interested in pseudo-Finsler spaces modeled on a Minkowski space which are not necessarily Berwald. We prove that most tensors of interest can be explicitly calculated. Namely, we shall show that they can be reduced to a form in which each term has a clear vertical or a horizontal interpretation. This very fact seem to have passed unnoticed, though it implies a dramatic simplification in the investigation of these spaces. Also it must be remarked that this family, though smaller than the general family of pseudo-Finsler spaces, is in the end fairly large. Most proposed metrics in applications are of this form, we mention the Berwald-Moór metric [15, 16], Bogoslovsky and Goenner's metric [17, 18], Bogoslovsky's metric [19, 20], the conic metric introduced by the second author in [21], or the Randers spaces in which the  $\beta$  1-form has constant  $\alpha$ -module.

One Finslerian quantity which attracted our interest is the Ricci scalar. In the theory of Finsler gravity most proposals for the vacuum dynamical equations imply the condition

$$Ric(y) = 0,$$

which is also suggested by studies of the Raychaudhuri equation [22, 13]. Although some of the final tensorial expressions which we shall present are rather concise, occasionally we had to pass through computations involving hundreds of terms. This happened precisely in the calculation of  $Ric(y)$  so we are happy to have been able to reduce its expression to a much simpler form, cf. Eq. (14). Here we have been helped by the *xAct* suite of tensor computer algebra [23].

## 2. The strategy

Let  $\{\tilde{e}_a\}$  be a basis of  $V$  and let  $y^a$  be the induced coordinates, so if  $\tilde{y} \in V$ , then there are  $y^a \in \mathbb{R}$ ,  $a = 1, \dots, m$ , such that  $\tilde{y} = y^a \tilde{e}_a$  (we adopt

the Einstein summation convention). Let us define at  $x \in M$ ,

$$y := \varphi_x^{-1}(\tilde{y}), \quad e_a := \varphi_x^{-1}(\tilde{e}_a)$$

so that  $y = y^a e_a$ .

Let us denote with  $\partial_\mu$  the basis of  $T_x M$  induced by a local coordinate system  $\{x^\mu\}$  on  $U \subset M$ , then there are invertible matrices (vierbein)  $e_a^\mu$  and  $e_\mu^a$ , such that  $e_a = e_a^\mu \partial_\mu$ ,  $e^a = e_\mu^a dx^\mu$ , where  $\{e^a\}$  is the dual basis to  $\{e_a\}$ ,

$$e_\mu^a e_b^\mu = \delta_b^a, \quad e_a^\mu e_\nu^\mu = \delta_\nu^a.$$

The vector  $y \in T_x M$  can be expanded as follows

$$y = y^\mu \partial_\mu = y^a e_a,$$

where (here we use the physicist convention according to which a different type of index, say Greek, Roman, might distinguish different objects)

$$y^\mu = y^a e_a^\mu.$$

We shall call  $y^a$  the *vertically induced internal variables* and  $y^\alpha$  the *horizontally induced internal variables*. The commutation coefficients  $[e_a, e_b] = c_{ab}^c e_c$  are

$$c_{ab}^c = e_a^\alpha e_b^\beta (e_{\alpha,\beta}^c - e_{\beta,\alpha}^c).$$

We shall also denote  $c_{ab,d}^c = c_{ab,\mu}^c e_d^\mu$ .

Now according to the isomorphism assumption between  $(T_x M, \mathcal{L}_x)$  and  $(V, L)$  we have

$$\mathcal{L}_x(y^a e_a) = L(y^a \tilde{e}_a)$$

In what follows  $\tilde{e}_a$  is considered chosen and fixed once and for all, so we can regard  $L$  as a function  $L: \mathbb{R}^m \rightarrow \mathbb{R}$ . So we denote

$$y_a = \partial_a L, \quad g_{ab} = \partial_a \partial_b L, \quad C_{abc} = \frac{1}{2} \partial_a \partial_b \partial_c L, \quad I_a = g^{bc} C_{abc}, \quad C_{abcd} = \partial_a C_{bcd}, \quad (2)$$

called respectively, the Legendre dual of  $y$ , the (pseudo-)Finsler metric, the Cartan torsion, the mean Cartan torsion and the Cartan curvature of the tangent pseudo-Minkowski space. Since the metric is non-degenerate we shall also have the inverse metric  $g^{ab}$ . To simplify notations we shall also

regard  $\mathcal{L}$  as a local function in the horizontally induced variables i.e. from  $U \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $U \subset \mathbb{R}^m$ , so that  $(x^\mu, y^\nu) \rightarrow \mathcal{L}(x^\mu, y^\nu)$ . Thus we can write

$$\mathcal{L}(x^\mu, y^\nu) = L(e_\nu^a(x)y^\nu).$$

Now, most references of Finsler geometry provide expressions of the most important tensors in horizontally induced coordinates so we shall stick to those [14, 24, 25, 26]. However, our aim is to change variables from the pair  $(x^\mu, y^\nu)$  to  $(x^\mu, y^a)$ . So we shall need the change of partial derivatives

$$\left. \frac{\partial}{\partial y^\alpha} \right)_{x^\gamma} = \left. \frac{\partial}{\partial y^a} \right)_{x^\gamma} e_\alpha^a, \quad (3)$$

$$\left. \frac{\partial}{\partial x^\alpha} \right)_{y^\gamma} = \left. \frac{\partial}{\partial x^\alpha} \right)_{y^c} + \left. \frac{\partial}{\partial y^b} \right)_{x^\gamma} e_{\beta, \alpha}^b e_c^\beta y^c. \quad (4)$$

The idea is to arrange the notable tensors in terms of the quantities  $e_\alpha^a$ ,  $e_a^\alpha$ ,  $c_{bc}^a$ ,  $c_{bc, d}^a$  which only depend on  $x^\mu$  and in terms of the quantities displayed in (2) including the inverse metric  $g^{ab}$  which only depend on  $y^a$ . So we do not raise or lower indices with the metric in the commutation coefficients otherwise we would spoil this property introducing objects dependent on both variables. Only after we have obtained all expressions of interest we simplify them lowering or raising the indices of the commutators, with the convention that the upper index in  $c_{bc}^a$  gets lowered to the left as  $c_{abc}$ . For instance applying the previous formulas to  $\mathcal{L}$  we get

$$y_\alpha := \frac{\partial \mathcal{L}}{\partial y^\alpha} = y_a e_\alpha^a, \\ \frac{\partial \mathcal{L}}{\partial x^\alpha} = y_b e_{\beta, \alpha}^b e_c^\beta y^c.$$

These derivatives enter the calculation of the spray

$$G^\mu := \frac{1}{2} g^{\mu\sigma} \left( \frac{\partial^2 \mathcal{L}}{\partial x^\rho \partial y^\sigma} y^\rho - \frac{\partial \mathcal{L}}{\partial x^\sigma} \right) \\ = e_m^\mu \left[ \frac{1}{2} (g^{mk} y_q y^n c_{kn}^q + e_{\delta, \rho}^m e_d^\delta e_p^\rho y^d y^p) \right].$$

Observe that all terms either depend on  $x^\mu$  or  $y^a$ ; this fact makes the next calculations easier by repeatedly using (3)-(4). However, we give the final expression in the shorter form

$$G^\mu = e_m^\mu \left[ \frac{1}{2} (y_q y^n c_{kn}^{qm} + e_{\delta, \rho}^m e_d^\delta e_p^\rho y^d y^p) \right], \quad (5)$$

which, we stress once again, it is not our working expression for the next calculations.

The coefficients of the non-linear connection are

$$N_\alpha^\mu := \frac{\partial G^\mu}{\partial y^\alpha} = e_l^\mu e_\alpha^b \left[ e_{\rho,\eta}^l e_b^\eta e_p^\rho y^p + c_{kn}^m (-C_b^{lk} y_m y^n + \frac{1}{2} (g^{lk} g_{mb} y^n + g^{lk} y_m \delta_b^n + \delta_m^l \delta_b^k y^n)) \right]. \quad (6)$$

The coefficients of the Berwald Finsler connection are

$$\begin{aligned} 2G_{\alpha\beta}^\rho &:= 2 \frac{\partial^2 G^\rho}{\partial y^\alpha \partial y^\beta} = [e_b^\rho (e_{\alpha,\beta}^b + e_{\beta,\alpha}^b)] \\ &+ c_{mn}^e \left( g^{rm} g_{ea} \delta_b^n + g^{rm} g_{eb} \delta_a^n + 2g^{rm} C_{eab} y^n - 2C_b^{rm} g_{ea} y^n \right. \\ &\left. - 2C_a^{rm} g_{eb} y^n - 2C_a^{rm} y_e \delta_b^n - 2C_b^{rm} y_e \delta_a^n - 2(\frac{\partial}{\partial y^b} C_a^{rm}) y_e y^n \right). \end{aligned} \quad (7)$$

We could simplify the vertical derivative with

$$-\frac{1}{2} \frac{\partial}{\partial y^b} \frac{\partial}{\partial y^a} g^{rm} = \frac{\partial}{\partial y^b} C_a^{rm} = C_{ab}^{rm} - 2C_b^{rs} C_{sa}^m - 2C_b^{ms} C_{sa}^r, \quad (8)$$

but the expression gets longer. The Berwald curvature tensor is defined through  $G_{\alpha\beta\gamma}^\rho = \frac{\partial^3 G^\rho}{\partial y^\alpha \partial y^\beta \partial y^\gamma}$  and in our case it is given by

$$2G_{abc}^r = 2G_{\alpha\beta\gamma}^\rho e_\rho^r e_\alpha^a e_\beta^b e_\gamma^c = c_{mn}^e \frac{\partial^3}{\partial y^a \partial y^b \partial y^c} (g^{rm} y_e y^n). \quad (9)$$

Here it is not convenient to expand the derivatives. The mean Berwald curvature  $E_{bc} := \frac{1}{2} G_{rbc}^r$  follows from

$$G_{rbc}^r = -c_{mn}^e \frac{\partial^2}{\partial y^b \partial y^c} (I^m y_e y^n). \quad (10)$$

The Douglas curvature is defined by  $D_{\alpha\beta\gamma}^\rho = \frac{\partial^3}{\partial y^\alpha \partial y^\beta \partial y^\gamma} (G^\rho - \frac{1}{m+1} N_a^\rho y^a)$  and in our case it is given by

$$D_{abc}^r = c_{pq}^e \frac{\partial^3}{\partial y^a \partial y^b \partial y^c} ((\frac{1}{2} g^{rp} + \frac{1}{m+1} I^p y^r) y_e y^q) \quad (11)$$

where  $m$  is the dimension of the Finsler space. In the next expression  $\nabla^{VC}$  is the vertical Cartan derivative (whose connection coefficients in coordinates  $\{y^a\}$  are  $C_{ab}^c$ ) and  $h_b^a := \delta_b^a - \frac{1}{g_y(y,y)} y^a y_b$ ,  $g_y(y,y) := 2L$  is the usual projection on the space tangent to the indicatrix.

The Landsberg tensor is

$$\begin{aligned}
L_{abc} &= L_{\alpha\beta\gamma} e_a^\alpha e_b^\beta e_c^\gamma = -\frac{1}{2} y_r G_{abc}^r = -\frac{1}{4} c_{mn}^e y_r \frac{\partial^3}{\partial y^a \partial y^b \partial y^c} (g^{rm} y_e y^n) \\
&= -\frac{1}{2} c_{mn}^e \left( y^m C_{ebc} \delta_a^n + y^m C_{eca} \delta_b^n + y^m C_{eab} \delta_c^n \right. \\
&\quad + C_{bc}^m g_{ea} y^n + C_{ca}^m g_{eb} y^n + C_{ab}^m g_{ec} y^n \\
&\quad + C_{bc}^m y_e \delta_a^n + C_{ca}^m y_e \delta_b^n + C_{ab}^m y_e \delta_c^n \\
&\quad \left. + 2(C_{abc}^m - C_{sa}^m C_{bc}^s - C_{sc}^m C_{ab}^s - C_{sb}^m C_{ca}^s) y_e y^n \right) \\
&= k_{mne} y^e (C_{bc}^m \delta_a^n + C_{ca}^m \delta_b^n + C_{ab}^m \delta_c^n) \\
&\quad + k_{mne} y^n y^e (C_{abc}^m - C_{bc}^s C_{as}^m - C_{ca}^s C_{bs}^m - C_{ab}^s C_{cs}^m). \\
&= k_{mne} y^e \left[ C_{bc}^{[m} h_a^{n]} + C_{ca}^{[m} h_b^{n]} + C_{ab}^{[m} h_c^{n]} + y^{[n} h^{m]s} h_a^p h_b^q h_c^r \nabla_s^{VC} C_{pqr} \right]
\end{aligned} \tag{12}$$

The first expressions are particularly suited for calculations since the whole  $x$  dependence is contained in the first  $c_{mn}^e$  factor, while the subsequent factors depend only on  $y^a$ . In the last two expressions we introduced the object anti-symmetric in the first two indices (they are sort of Ricci rotation coefficients)

$$k_{mne}(x, y) = \frac{1}{2} (c_{mne} + c_{nem} - c_{emn}).$$

The last expression in Eq. (12) is useful because through contraction with  $y_n$  it shows that for  $m \geq 3$  the tensor in square bracket vanishes if and only if  $C_{abc} = 0$ . Thus it is not possible to find vertical conditions weaker than  $C_{abc} = 0$  (i.e. the pseudo-Riemannian case) which guarantee that  $L_{abc} = 0$ .

Still the expression for  $L$  simplifies considerably when  $h_a^p h_b^q h_c^r h_d^s \nabla_s^{VC} C_{pqr} = 0$  which happens if and only if the geometry of the indicatrix is such that  $\nabla^h \mathbf{c} = 0$  where  $\mathbf{c}$  is the Pick cubic form and  $\mathbf{h}$  is the affine metric for the centroaffine transverse (for translation between Finsler and centroaffine geometry the reader is referred to [27]). For instance,  $\nabla^h \mathbf{c} = 0$  holds true whenever the indicatrix is a homogeneous affine sphere [28, 29].

The mean Landsberg curvature is

$$\begin{aligned}
J_c &:= L_{ac}^a = -\frac{1}{2} c_{mn}^e \left( y^m I_e \delta_c^n + I^m g_{ec} y^n + I^m y_e \delta_c^n + 2 \left( \frac{\partial I^m}{\partial y^c} + C_{sc}^m I^s \right) y_e y^n \right) \\
&= k_{mne} y^e \left[ I^m \delta_c^n + y^n \left( \frac{\partial I^m}{\partial y^c} + C_{sc}^m I^s \right) \right] \\
&= k_{mne} y^e \left[ h_c^{[n} I^{m]} + y^{[n} (h_c^a h_b^{m]} \nabla_a^{VC} I^b) \right].
\end{aligned} \tag{13}$$



Again, the first expression is useful in calculations since it factors the  $x^\mu$  and  $y^a$  dependence.

The last formula can be used to show that the vertical tensor in square brackets vanishes if and only if  $I = 0$  (for the ‘only if’ direction contract first with  $y_n$ ). Thus it is not possible to find vertical conditions weaker than  $I_c = 0$  which guarantee that  $J_c = 0$  (a fact known to hold in general for any pseudo-Finsler space).

The curvature of the non-linear connection is

$$R_{\alpha\beta}^\mu := \frac{\delta N_\beta^\mu}{\delta x^\alpha} - \frac{\delta N_\alpha^\mu}{\delta x^\beta}, \quad \frac{\delta}{\delta x^\alpha} := \frac{\partial}{\partial x^\alpha} - N_\alpha^\mu \frac{\partial}{\partial y^\mu}$$

Our calculation for the Ricci scalar gives

$$\begin{aligned} Ric(y) &:= R_{\mu\alpha}^\mu y^\alpha = 2 \frac{\partial G^\mu}{\partial x^\mu} - y^\mu \frac{\partial^2 G^\nu}{\partial x^\mu \partial y^\nu} + 2G^\mu \frac{\partial^2 G^\nu}{\partial y^\mu \partial y^\nu} - \frac{\partial G^\mu}{\partial y^\nu} \frac{\partial G^\nu}{\partial y^\mu} \\ &= g^{ad} y^b y_c c_{ab,d}^c + y^a y^b c_{bc,a}^c + I^b y^a y^c y_d c_{bc,a}^d \\ &\quad + c_{jk}^i c_{mn}^l g_{ia} g_{lb} \left( \frac{1}{4} g^{jm} g^{kn} y^a y^b + g^{bn} g^{km} y^a y^j - \frac{1}{2} g^{an} g^{bk} y^j y^m \right) \\ &\quad - \frac{1}{2} g^{ab} g^{kn} y^j y^m + 2y^a y^j y^m C^{bkn} - y^a y^j y^b y^m C_h^{kg} C_g^{mh} \\ &\quad + I^j g^{kn} y^a y^b y^m - g^{bk} I^n y^a y^j y^m - g^{en} y^a y^j y^b y^m \frac{\partial I^k}{\partial y^e} \Big). \end{aligned} \quad (14)$$

It can be easily checked that if the metric does not depend on the vertical variable ( $C_{abc} = 0$ ) we get the usual expression of the Ricci tensor in terms of the commutation coefficients [30, Sect. 98].

### 3. Conclusions

We have shown that for pseudo-Finsler spaces modeled on a pseudo-Minkowski space several Finslerian quantities can be explicit calculated. We have found interesting expressions for the Berwald curvature, Douglas curvature, Landsberg tensor, mean Landsberg tensor, mean Berwald curvature, and Ricci scalar in terms of: (a) vertical quantities, which only depend on the geometry of the indicatrix, and (b) horizontal quantities, namely the commutation coefficients, which only depend on how the model Minkowski space is displaced all over the manifold. These different contributions can be separately investigated for specific metrics. We hope that our results could serve to bridge the gap between abstract Finsler geometry and its applications.

## Acknowledgements

AGP wishes to thank the “Dipartimento di Matematica e Informatica U. Dini” at Florencia University where this work was carried out for hospitality and financial support. AGP is supported by the projects FIS2014-57956-P of Spanish “Ministerio de Economía y Competitividad” and PTDC/MAT-ANA/1275/2014 of Portuguese “Fundação para a Ciência e a Tecnologia”.

## References

- [1] H. Martini, K. J. Swanepoel, G. Weiß, The geometry of Minkowski spaces—a survey. I, *Expo. Math.* 19 (2) (2001) 97–142.
- [2] J. C. Álvarez Paiva, A. C. Thompson, Volumes on normed and Finsler spaces, in: *A sampler of Riemann-Finsler geometry*, Vol. 50 of *Math. Sci. Res. Inst. Publ.*, Cambridge Univ. Press, Cambridge, 2004, pp. 1–48.
- [3] Y. Ichijō, Finsler manifolds modeled on a Minkowski space, *J. Math. Kyoto Univ.* 16 (3) (1976) 639–652.
- [4] T. Aikou, Some remarks on Finsler vector bundles, *Publ. Math. Debrecen* 57 (3-4) (2000) 367–373.
- [5] M. Matsumoto, On Finsler spaces with 1-form metric, *Tensor N.S.* 32 (1978) 161–169.
- [6] H. Izumi, Nonholonomic frames in a Finsler space with a 1-form metric, *Tensor (N.S.)* 40 (2) (1983) 189–192.
- [7] T. Sakaguchi, Remarks on Finsler spaces with 1-form metric, *Tensor (N.S.)* 40 (2) (1983) 173–183.
- [8] E. Asanov, G.S.; Kirnasov, On 1-form Finsler spaces, *Rep. Math. Phys.* 19.
- [9] J. Szilasi, L. Tamássy, Generalized Berwald spaces as affine deformations of Minkowski spaces, *Rev. Roumaine Math. Pures Appl.* 57 (2) (2012) 165–178.
- [10] L. Huang, On the fundamental equations of homogeneous Finsler spaces, *Differential Geom. Appl.* 40 (2015) 187–208.

- [11] Z. I. Szabó, Positive definite Berwald spaces. Structure theorems on Berwald spaces, *Tensor (N.S.)* 35 (1) (1981) 25–39.
- [12] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, 2000.
- [13] E. Minguzzi, Raychaudhuri equation and singularity theorems in Finsler spacetimes, *Class. Quantum Grav.* 32 (2015) 185008, arXiv:1502.02313.
- [14] E. Minguzzi, The connections of pseudo-Finsler spaces, *Int. J. Geom. Meth. Mod. Phys.* 11 (2014) 1460025, erratum *ibid* 12 (2015) 1592001. arXiv:1405.0645.
- [15] L. Berwald, Über Finslersche und Cartansche Geometrie II. Invarianten bei der Variation vielfacher Integrale und Parallelhyperflächen in Cartanschen Räumen, *Compositio Math.* 7 (1939) 141–176.
- [16] A. Moór, Ergänzung zu meiner Arbeit: “Über die Dualität von Finslerschen und Cartanschen Räumen.”, *Acta Math.* 91 (1954) 187–188.
- [17] G. Y. Bogoslovsky, H. F. Goenner, On a possibility of phase transitions in the geometric structure of space-time, *Physics Letters A* 244 (1998) 222–228.
- [18] G. Y. Bogoslovsky, H. F. Goenner, Finslerian spaces possessing local relativistic symmetry, *Gen. Relativ. Gravit.* 31 (1999) 1565–1603.
- [19] G. Y. Bogoslovsky, A special-relativistic theory of the locally anisotropic space-time. I: The metric and group of motions of the anisotropic space of events, *Il Nuovo Cimento* 40 B (1977) 99–115.
- [20] G. Y. Bogoslovsky, A viable model of locally anisotropic space-time and the Finslerian generalization of the relativity theory, *Fortschr. Phys.* 42 (1994) 143–193.
- [21] E. Minguzzi, Affine sphere spacetimes which satisfy the relativity principle, *Phys. Rev. D* In press.
- [22] S. F. Rutz, A Finsler generalisation of Einstein’s vacuum field equations, *Gen. Relativ. Gravit.* 25 (1993) 1139–1158.

- [23] J. M. Martín-García, *xAct: efficient tensor computer algebra*, <http://www.xact.es>.
- [24] A. Bejancu, H. R. Farran, *Geometry of pseudo-Finsler submanifolds*, Kluwer Academic Publishers, Dordrecht, 2000.
- [25] Z. Shen, *Lectures on Finsler geometry*, World Scientific, Singapore, 2001.
- [26] J. Szilasi, R. L. Lovas, D. C. Kertesz, *Connections, sprays and Finsler structures*, World Scientific, London, 2014.
- [27] E. Minguzzi, *Affine sphere relativity* Comm. Math. Phys. DOI:10.1007/s00220-016-2802-9.
- [28] Z. Hu, H. Li, L. Vrancken, *Locally strongly convex affine hypersurfaces with parallel cubic form*, J. Differential Geom. 87 (2) (2011) 239–308.
- [29] R. Hildebrand, *Centro-affine hypersurface immersions with parallel cubic form*, Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry 56 (2) (2015) 593–640.
- [30] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley Publishing Company, Reading, 1962.